

# Recognizing Sequences

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Here's an IQ-test question:

What is the next number in the sequence 1, 3, 6, 10, ...?

The purpose of this section of the course is to develop the mental habits that make solving this type of question easier. Along the way, we'll see the golden ratio (have you read the Da Vinci Code yet?), and you'll learn the essence of calculus. The practice of calculus is harder, and you should enroll in Math 231 to absorb it, but the essence of the most important thought ever had by man can be understood already.

## 1 Lines

When confronted with a sequence, the first step is to look at the differences between consecutive elements. Here's a sequence:

1, 3, 5, 7, 9, ...

The difference between consecutive elements is 2. Always 2. As in,  $2 = 3 - 1 = 5 - 3 = 7 - 5 = 9 - 7 = \dots$ . How can we use this? Philosophically, we know that the sequence begins with 1, and it always changes by 2: if you know where something is, and you know which direction it's going, then you should be able to figure out where it will end up!

How does the math play out? We have the sequence of differences 2, 2, 2, 2, ..., which we can summarize with the formula  $b_n = 2$ . From this, we get  $a_n = 2n + c$ , for some constant  $c$ . The "2" comes from  $b_n$  being that constant. And  $c$ ? We know that  $a_1 = 1$ , the first term in the sequence. And

from our formula we know that  $a_1 = 2 \cdot 1 + c$ . Reconciling these two descriptions of  $a_1$  gives the equation  $1 = 2 \cdot 1 + c$ , whence  $c = -1$ . The formula for the sequence is  $a_n = 2n - 1$ .

Here's another:

$$7, 1, -5, -11, -17, \dots$$

The sequence of differences is

$$-6, -6, -6, -6, \dots$$

The formula for the differences is  $b_n = -6$ , so the formula for the sequence is  $a_n = -6n + c$ . What's  $c$ ? We know that  $a_1 = 7$ , and the formula says  $a_1 = -6 \cdot 1 + c$ . Thus,  $c = 13$ . The formula is

$$a_n = -6n + 13.$$

Here's another

$$31, 34, 37, 40, \dots$$

The sequence of differences is

$$3, 3, 3, 3, \dots$$

The formula for the differences is  $b_n = 3$ , so the formula for the sequence is  $a_n = 3n + c$ . What's  $c$ ? We know that  $a_1 = 31$ , and the formula says  $a_1 = 3 \cdot 1 + c$ . Thus,  $c = 28$ .

**The moral:** If the sequence of differences is constant  $A$ , then the formula for the sequence is  $An + c$  for some  $c$ , and you need to use the first term of the sequence to get  $c$ .

## 2 Parabolas

Here's another problem:

$$0, 1, 4, 9, 16, \dots$$

Do you recognize the sequence? Can you write down a formula for the  $n$ -th term (0 is the first term)? Let's look at the sequence of differences:

$$1, 3, 5, 7, \dots$$

That's look familiar! We get a formula for this sequence as above. These differences are given by the formula  $b_n = 2n - 1$ . From here, things get tricky. If the differences are given by  $An + B$ , then the sequence is given by

$$\frac{A}{2}n^2 + (B - A/2)n + c,$$

for some constant  $c$ . In this case, we have  $A = 2$  and  $B = -1$ , so  $a_n = \frac{2}{2}n^2 + (-1 - 2/2)n + c = n^2 - 2n + c$ . What's  $c$ ? We have  $a_1 = 0$  from the sequence, and  $a_1 = 1^2 - 2 \cdot 1 + c$  from the formula. Thus,  $0 = 1 - 2 + c$ , whence  $c = 1$ . The formula is

$$a_n = n^2 - 2n + 1.$$

Here's another problem of this sort. What's the 30-th term of the sequence that begins

$$0, 5, 16, 33, 56, 85, 120, \dots?$$

The sequence of differences is

$$5, 11, 17, 23, 29, 35, \dots$$

The sequence of differences of this is

$$6, 6, 6, 6, \dots$$

So we have  $c_n = 6$ , and  $b_n = 6n + c$  for some constant  $c$ . We're using the symbol  $c$  for two different things? Yes, but that's life. We get  $5 = b_1 = 6 \cdot 1 + c$ , whence  $c = -1$  and  $b_n = 6n - 1$ . Now  $a_n = \frac{6}{2}n^2 + (-1 - 6/2)n + c = 3n^2 - 4n + c$ . We have

$$0 = a_1 = 3 \cdot 1^2 - 4 \cdot 1 + c$$

whence  $c = 1$ . We get the formula for the original sequence:

$$a_n = 3n^2 - 4n + 1.$$

The 30-th term is  $a_{30} = 3(30)^2 - 4(30) + 1 = 3(900) - 120 + 1 = 2581$ .

**The moral:** If the sequence of differences is constant  $A$ , then the sequence is given by  $An + c$  for some  $c$ . If the sequence of differences is given by  $An + B$ , then the sequence is given by  $\frac{A}{2}n^2 + (B - A/2)n + c$  for some constant  $c$ . In both cases, you use the first term of the sequence to find  $c$ .

For each of the sequences below, find the next couple of terms in the sequence, and give a formula for the  $n$ -th term (the first term listed is  $a_1$ ).

• 3, 3, 3, 3, 3, ...

• 1, 2, 3, 4, 5, ...

• 3, 4, 5, 6, ...

• 3, 2, 1, 0, -1, ...

• 0, 1, 4, 9, 16, ...

• 1, 2, 4, 7, 11, ...

• 9, 16, 25, 36, 49, ...

• 2, 6, 12, 20, 30, ...

• 1, 5, 9, 13, 17, ...

• 2, 7, 14, 23, 34, ...

- 2, 4, 8, 16, 32, ...
- 3, 11, 19, 27, 35, ...
- 0, 6, 24, 60, 120, 210, 336, ...
- -3, -1, 1, 3, 5, ...
- 3, 7, 12, 19, 28, ...
- Find a polynomial that gives a sequence beginning 17, 691, 13, ...